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SUPPLEMENT TO CLASSIFICATION OF THREE-FOLD DIVISORIAL CONTRACTIONS

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ABSTRACT. Every three-fold divisorial contraction to a non-Gorenstein point is a weighted blow-up.

This supplement finishes the explicit description of a three-fold divisorial contraction whose exceptional divisor contracts to a non-Gorenstein point. Contractions to a quotient singularity were treated by Kawamata in [8]. The author's study [7], based on the singular Riemann–Roch formula, provided the classification except for the case of small discrepancy. On the other hand, Hayakawa classified those with discrepancy at most one in [1], [2], [3], by the fact that there exist only a finite number of divisors with such discrepancy over a fixed singularity. The only case left was when it is a contraction to a c$D/2$ point with discrepancy two. We demonstrate its classification in Theorem 2 by the method in [7]. It turns out that every contraction is a weighted blow-up.

Theorem 1. *Let $f\colon Y \to X$ be a three-fold divisorial contraction whose exceptional divisor E contracts to a non-Gorenstein point P. Then f is a weighted blow-up of the singularity $P \in X$ embedded into a cyclic quotient of a smooth five-fold.*

Our method of the classification is to study the structure of the bi-graded ring $\bigoplus_{i,j} f_*\mathscr{O}_Y(iK_Y + jE)/f_*\mathscr{O}_Y(iK_Y + jE - E)$. We find local coordinates at P to meet this structure and verify that f should be a certain weighted blow-up. The choice of local coordinates is restricted by the action of the cyclic group, which makes easier the classification in the non-Gorenstein case. We do not know if this method is sufficient to settle all the remaining Gorenstein cases in [4], [5], [6] with discrepancy at most four.

By a three-fold divisorial contraction to a point, we mean a projective morphism $f\colon (Y \supset E) \to (X \ni P)$ between terminal three-folds such that $-K_Y$ is f-ample and the exceptional locus E is a prime divisor contracting to a point P. We shall treat f on the germ at P in the complex analytic category. According to [7, Theorems 1.2, 1.3], the only case left is

type e1 with $P = \mathrm{c}D/2$, the discrepancy $a/n = 4/2$

in [7, Table 3]. We shall prove the following theorem.

Theorem 2. *Suppose that f is a divisorial contraction of type* e1 *to a c$D/2$ point with discrepancy* 2. *Then f is the weighted blow-up with* $\mathrm{wt}(x_1,x_2,x_3,x_4,x_5) = (\frac{r+1}{2}, \frac{r-1}{2}, 2, 1, r)$ *with $r \geq 7$, $r \equiv \pm 1$ mod 8 for a suitable identification*

$$P \in X \;\simeq\; o \in \begin{pmatrix} x_1^2 + x_4x_5 + p(x_2,x_3,x_4) = 0 \\ x_2^2 + q(x_1,x_3,x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5_{x_1x_2x_3x_4x_5}/\frac{1}{2}(1,1,1,0,0),$$

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such that p is of weighted order $> r$ and q is weighted homogeneous of weight $r-1$ for the weights distributed above.

The proof is along the argument in [7, Section 7]. Henceforth $f\colon (Y \supset E) \to (X \ni P)$ is a divisorial contraction of type e1 to a c$D/2$ point with discrepancy 2. By [7, Table 3], Y has only one singular point Q say at which E is not Cartier. Q is a quotient singularity of type $\frac{1}{2r}(1,-1,r+4)$ with $r \ge 7$, $r \equiv \pm 1 \bmod 8$.

We set vector spaces $V_i = V_i^0 \oplus V_i^1$ with

$$V_i^0 := f_*\mathscr{O}_Y(-iE)/f_*\mathscr{O}_Y(-(i+1)E),$$
$$V_i^1 := f_*\mathscr{O}_Y(K_Y-(i+2)E)/f_*\mathscr{O}_Y(K_Y-(i+3)E).$$

They are zero for negative i, and we have the (bi-)graded ring $\bigoplus V_i$ by a local isomorphism $\mathscr{O}_X(2K_X) \simeq \mathscr{O}_X$. To study its structure in lower-degree part, we first compute the dimensions of V_i^j in terms of the finite sets

$$N_i := \Big\{(l_1,l_2,l_3,l_4,l_5) \in \mathbb{Z}^5_{\ge 0} \;\Big|\; \frac{r+1}{2}l_1 + \frac{r-1}{2}l_2 + 2l_3 + l_4 + rl_5 = i,\ \ l_1,l_2 \le 1\Big\}.$$

N_i is decomposed into $N_i^0 \sqcup N_i^1$ with $N_i^j := \{(l_1,l_2,l_3,l_4,l_5) \in N_i \mid l_1+l_2+l_3 \equiv j \bmod 2\}$.

Lemma 3. $\dim V_i^j = \# N_i^j$.

Proof. We follow the notation in [7]. $(r_Q, b_Q, v_Q) = (2r, r+4, 2)$ and $E^3 = 1/r$ by [7, Tables 2, 3]. By $\dim V_i^j = d(j, -i-2j)$ for $i \ge -2$ in [7, (2.8)], the equality [7, (2.6)] for $(j, -i-2j)$ implies that for $i \ge 0$,

$$\dim V_i^j - \dim V_{i-2}^{1-j} = \frac{2i+1}{r} + B_{2r}(2i+rj+2) - B_{2r}(2i+rj).$$

Here $B_{2r}(k) = (\overline{k} \cdot \overline{2r-k})/2r$ and $\bar{}$ denotes the residue modulo $2r$. On the other hand, by $N_i^j = \big(N_{i-2}^{1-j} + (0,0,1,0,0)\big) \sqcup \{(l_1,l_2,0,l_4,l_5) \in N_i^j\}$,

$$\#N_i^j - \#N_{i-2}^{1-j} = \begin{cases} \#\{(0,0,0,l_4,l_5) \in N_i^0\} + \#\{(1,1,0,l_4,l_5) \in N_i^0\} & \text{for } j=0, \\ \#\{(0,1,0,l_4,l_5) \in N_i^1\} + \#\{(1,0,0,l_4,l_5) \in N_i^1\} & \text{for } j=1. \end{cases}$$

The lemma follows by verifying the coincidence of their right-hand sides. q.e.d.

We shall find bases of V_i starting with an arbitrary identification

$$P \in X \;\simeq\; o \in (\phi = 0) \subset \mathbb{C}^4_{x_1x_2x_3x_4}/\frac{1}{2}(1,1,1,0). \tag{1}$$

For a semi-invariant function h, $\operatorname{ord}_E h$ denotes the order of h along E.

Lemma 4. (i) $\operatorname{ord}_E x_4 = 1$ *and* $\operatorname{ord}_E x_i \ge 2$ *for* $i = 1,2,3$. *There exists some* k *with* $\operatorname{ord}_E x_k = 2$. *We set* $x_k = x_3$ *by permutation.*

(ii) *For* $i < \frac{r-1}{2}$, *the monomials* $x_3^{l_3}x_4^{l_4}$ *for* $(0,0,l_3,l_4,0) \in N_i$ *form a basis of* V_i. *In particular for* $k = 1,2$, $\operatorname{ord}_E \bar{x}_k \ge \frac{r-1}{2}$ *for* $\bar{x}_k := x_k + \sum c_{kl_3l_4}x_3^{l_3}x_4^{l_4}$ *with some* $c_{kl_3l_4} \in \mathbb{C}$, *with summation over* $(0,0,l_3,l_4,0) \in \bigcup_{i<\frac{r-1}{2}} N_i^1$.

(iii) *There exists some* k *with* $\operatorname{ord}_E \bar{x}_k = \frac{r-1}{2}$ *such that the monomials* $\bar{x}_k$ *and* $x_3^{l_3}x_4^{l_4}$ *for* $(0,0,l_3,l_4,0) \in N_{\frac{r-1}{2}}$ *form a basis of* $V_{\frac{r-1}{2}}$. *We set* $\bar{x}_k = \bar{x}_2$ *by permutation, then* $\operatorname{ord}_E \hat{x}_1 \ge \frac{r+1}{2}$ *for* $\hat{x}_1 := \bar{x}_1 + \sum c_{l_2l_3l_4}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ *with some* $c_{l_2l_3l_4} \in \mathbb{C}$, *with summation over* $(0,l_2,l_3,l_4,0) \in N^1_{\frac{r-1}{2}}$.

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(iv) $\mathrm{ord}_E\,\hat{x}_1=\frac{r+1}{2}$. *For $i<r-1$, the monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4,0)\in N_i$ form a basis of V_i.*

(v) *Set $\tilde{N}_i:=\{(l_1,l_2,l_3,l_4,l_5)\in\mathbb{Z}^5_{\geq 0}\mid \frac{r+1}{2}l_1+\frac{r-1}{2}l_2+2l_3+l_4+rl_5=i\}$ and $\tilde{N}_i^0:=\{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_i\mid l_1+l_2+l_3 \text{ even}\}$. The monomials $\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}$ for $(l_1,l_2,l_3,l_4,0)\in\tilde{N}^0_{r-1}$ have one non-trivial relation, say ψ, in V^0_{r-1}. The natural exact sequence below is exact.*

$$0\to\mathbb{C}\psi\to\bigoplus_{(l_1,l_2,l_3,l_4,0)\in\tilde{N}_{r-1}}\mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\to V_{r-1}\to 0.$$

(vi) $\mathrm{ord}_E\,\psi=r$. *The natural exact sequence below is exact.*

$$0\to\mathbb{C}x_4\psi\to\bigoplus_{(l_1,l_2,l_3,l_4,l_5)\in\tilde{N}_r}\mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\psi^{l_5}\to V_r\to 0.$$

Proof. We follow the proof of [7, Lemma 7.2], with using the computation of Lemma 3. (i) follows from $\dim V_1^0=1$, $\dim V_1^1=0$ and $\dim V_2^1=1$. Then V_4^0 is spanned by x_3^2 and x_4^4, which should form a basis of V_4^0 by $\dim V_4^0=2$. Now (ii) to (v) follow from the same argument as in [7, Lemma 7.2]. We obtain the sequence in (vi) also, which is exact possibly except for the middle. Its exactness is verified by comparing dimensions. q.e.d.

Corollary 5. *We distribute weights* $\mathrm{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4)=(\frac{r+1}{2},\frac{r-1}{2},2,1)$ *to the coordinates $\hat{x}_1,\bar{x}_2,x_3,x_4$ obtained in Lemma* 4. *Then ϕ in* (1) *is of form*

$$\phi=cx_4\psi+\phi_{>r}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

with $c\in\mathbb{C}$ and a function $\phi_{>r}$ of weighted order $>r$, where ψ is as in Lemma 4(v).

Proof. Decompose $\phi=\phi_{\leq r}+\phi_{>r}$ into the part $\phi_{\leq r}$ of weighted order $\leq r$ and $\phi_{>r}$ of weighted order $>r$. Then $\mathrm{ord}_E\,\phi_{\leq r}=\mathrm{ord}_E\,\phi_{>r}>r$, so $\phi_{\leq r}$ is mapped to zero by the natural homomorphism

$$\bigoplus_{(l_1,l_2,l_3,l_4,0)\in\bigcup_{i\leq r}\tilde{N}_i^0}\mathbb{C}\hat{x}_1^{l_1}\bar{x}_2^{l_2}x_3^{l_3}x_4^{l_4}\to\mathscr{O}_X/f_*\mathscr{O}_Y(-(r+1)E),$$

whose kernel is $\mathbb{C}x_4\psi$ by Lemma 4(iv)-(vi). q.e.d.

We shall derive an expression of the germ $P\in X$ in Theorem 2. By [9, Remark 23.1], the c$D/2$ point $P\in X$ has an identification in (1) with ϕ either of

(A) $$\phi=x_1^2+x_2x_3x_4+x_2^{2\alpha}+x_3^{2\beta}+x_4^{\gamma},$$

(B) $$\phi=x_1^2+x_2^2x_4+\lambda x_2x_3^{2\alpha-1}+g(x_3^2,x_4),$$

with $\alpha,\beta\geq 2$, $\gamma\geq 3$, $\lambda\in\mathbb{C}$ and $g\in(x_3^4,x_3^2x_4^2,x_4^3)$. As its general elephant has type D_k with $k\geq 2r$ by [7, Lemma 5.2(i)], we have

(2) $$\gamma\geq r\ \text{ in (A)},\qquad \mathrm{ord}\,g(0,x_4)\geq r\ \text{ in (B)}.$$

Lemma 6. *The case* (A) *does not happen.*

Proof. By Lemma 4(i), $\mathrm{ord}_E\,x_4=1$, $\mathrm{ord}_E\,x_i\geq 2$ for $i=1,2,3$ and some $\mathrm{ord}_E\,x_i=2$. $\mathrm{ord}_E\,x_1\geq 3$ by the relation $-x_1^2=x_2x_3x_4+x_2^{2\alpha}+x_3^{2\beta}+x_4^{\gamma}$ and (2). Thus we may set $\mathrm{ord}_E\,x_3=2$ by permutation, and construct $\bar{x}_1,\bar{x}_2$ as in Lemma 4(ii).

Let $W_{\frac{r-1}{2}}$ be the subspace of $V_{\frac{r-1}{2}}$ spanned by the monomials in x_3,x_4. If $\bar{x}_1\notin W_{\frac{r-1}{2}}$, the triple $(\bar{x}_1,x_3,x_4)$ plays the role of $(\bar{x}_2,x_3,x_4)$ in Lemma 4(iii). We construct $\hat{x}_2$ as in Lemma 4(iii) to obtain a quartuple $(\hat{x}_2,\bar{x}_1,x_3,x_4)$, and distribute

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$\mathrm{wt}(\hat{x}_2,\bar{x}_1,x_3,x_4)=(\frac{r+1}{2},\frac{r-1}{2},2,1)$ as in Corollary 5. Set $\bar{x}_1=x_1+p_1(x_3,x_4)$, $\hat{x}_2=x_2+p_2(\bar{x}_1,x_3,x_4)$ and rewrite ϕ as

$$\phi=(\bar{x}_1-p_1)^2+(\hat{x}_2-p_2)x_3x_4+(\hat{x}_2-p_2)^{2\alpha}+x_3^{2\beta}+x_4^{\gamma}.$$

ϕ has the term $\bar{x}_1^2$ of weight $r-1$, which contradicts Corollary 5.

Hence $\bar{x}_1\in W_{\frac{r-1}{2}}$, and we obtain a quartuple $(\hat{x}_1,\bar{x}_2,x_3,x_4)$ by $\hat{x}_1=x_1+p_1(x_3,x_4)$, $\bar{x}_2=x_2+p_2(x_3,x_4)$ as in Lemma 4. Distribute $\mathrm{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4)=(\frac{r+1}{2},\frac{r-1}{2},2,1)$ and rewrite ϕ as

$$\phi=(\hat{x}_1-p_1)^2+(\bar{x}_2-p_2)x_3x_4+(\bar{x}_2-p_2)^{2\alpha}+x_3^{2\beta}+x_4^{\gamma}.$$

ϕ has the term $\bar{x}_2x_3x_4$ of weight $\frac{r+5}{2}$, whence $\frac{r+5}{2}\geq r$ by Corollary 5, a contradiction to $r\geq 7$. q.e.d.

Lemma 7. *The germ $P\in X$ has an expression in Theorem 2, with q not of form $(x_3s(x_3^2,x_4))^2$, such that each $\mathrm{ord}_E\,x_i$ coincides with $\mathrm{wt}\,x_i$ distributed in Theorem 2.*

Proof. We have the case (B) by Lemma 6. $\mathrm{ord}_E\,x_4=1$ and $\mathrm{ord}_E\,x_1\geq 3$ as in (A), then $\mathrm{ord}_E\,x_2\geq 3$ and $\mathrm{ord}_E\,x_3=2$. We construct $\bar{x}_1,\bar{x}_2$ as in Lemma 4(ii). By the same reason as in the proof of Lemma 6, we obtain $\bar{x}_1\in W_{\frac{r-1}{2}}$ and a quartuple $(\hat{x}_1,\bar{x}_2,x_3,x_4)$ by $\hat{x}_1=x_1+p_1(x_3,x_4)$, $\bar{x}_2=x_2+p_2(x_3,x_4)$. Distribute $\mathrm{wt}(\hat{x}_1,\bar{x}_2,x_3,x_4)=(\frac{r+1}{2},\frac{r-1}{2},2,1)$ and rewrite ϕ as

$$\phi=(\hat{x}_1-p_1)^2+(\bar{x}_2-p_2)^2x_4+\lambda(\bar{x}_2-p_2)x_3^{2\alpha-1}+g(x_3^2,x_4).$$

ϕ has the term $\bar{x}_2^2x_4$ of weight r and should be of form

$$\phi=(\bar{x}_2^2+h(\hat{x}_1,\bar{x}_2,x_3,x_4))x_4+\phi_{>r}(\hat{x}_1,\bar{x}_2,x_3,x_4)$$

as in Corollary 5 with $\psi=\bar{x}_2^2+h(\hat{x}_1,\bar{x}_2,x_3,x_4)$. In particular $p_2=0$ as otherwise $p_2\bar{x}_2x_4$ would be of weighted order $<r$, and one can write

$$\phi=\hat{x}_1^2+x_4\psi+p(\bar{x}_2,x_3,x_4),\qquad \psi=\bar{x}_2^2+q(\hat{x}_1,x_3,x_4),$$

where p is of weighted order $>r$ and q is weighted homogeneous of weight $r-1$. A desired expression is derived by setting $x_5:=-\psi$ and replacing x_4 with $-x_4$. q is not of form $(x_3s(x_3^2,x_4))^2$ by Lemma 4(iii) and $\mathrm{ord}_E(\bar{x}_2^2+q)=r$. q.e.d.

Take an expression of the germ $P\in X$ in Theorem 2 by Lemma 7. We shall apply the extension of [7, Lemma 6.1] to the case when X is embedded into a cyclic quotient of $\mathbb{C}^5$. Let $g\colon Z\to X$ be the weighted blow-up with $\mathrm{wt}\,x_i=\mathrm{ord}_E\,x_i$. By direct calculation, we verify the assumptions of [7, Lemma 6.1] and that Z is smooth outside the strict transform of $x_1x_2x_3x_4x_5=0$. We need the condition $q\neq(x_3s)^2$ to check that the restriction $\bar{F}\cap Z$ of the exceptional locus in the ambient space defines an irreducible reduced 2-cycle on Z. Therefore f should coincide with g by [7, Lemma 6.1], and Theorem 2 is completed.

Remark 8. Using $H\cap E\simeq\mathbb{P}^1$ in the proof of [7, Theorem 5.4], one can show that

(i) if $r\equiv 1$ mod 8, $x_2x_3^{(r+3)/4}$ appears in p and $x_3^{(r-1)/2}$ appears in q,

(ii) if $r\equiv 7$ mod 8, $x_3^{(r+1)/2}$ appears in p and $x_1x_3^{(r-3)/4}$ appears in q.

Theorem 1 follows from [1], [2], [3], [7], [8] and Theorem 2.

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